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The exact calculation of a gravity multi-matrix superpropagator

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Abstract. A particular multi-matrix superpropagator, occurring in exponentially parametrized quantum gravity, is calculated in detail. It is shown that the method used can be applied to calculate a whole set of multi-matrix superpropagators. Furthermore it is demonstrated that gauges exist in which the superpropagators belonging to the set behave well asymptotically.

1. Introduction

A powerful technique has been developed by Ashmore and Delbourgo (1973) to calculate matrix superpropagators occurring in non-polynomial Lagrangian field theories. As an example they have demonstrated that the exponentially parametrized gravity superpropagator

$$\langle |-g(x)|^{\omega} g_{\alpha\beta}(x), |-g(0)|^{\omega} g_{\gamma\delta}(0)\rangle \tag{1.1}$$

with

$$|-g(x)|^{\omega} g_{\mu\nu}(x) = [\exp \kappa\phi(x)]_{\mu\nu} \tag{1.2}$$

can be calculated in closed form for all values of the weight parameter ω. Given the free propagator of the matrix field $\phi_{\alpha\beta}$ in the form

$$\langle \phi_{\alpha\beta}(x), \phi_{\gamma\delta}(0)\rangle = \tfrac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - 2c\eta_{\alpha\beta}\eta_{\gamma\delta})\,\Delta(x), \tag{1.3}$$

where $\eta_{\alpha\beta}$ is the Minkowski tensor, $\Delta(x)$ is the free propagator for the massless scalar field and c is a gauge parameter, they first derive an integral representation for the matrix superpropagator

$$\langle \phi^N_{\alpha\beta}(x), \phi^N_{\gamma\delta}(0)\rangle$$

for arbitrary N. This superpropagator, written most generally in the form

$$\langle \phi^N_{\alpha\beta}(x), \phi^N_{\gamma\delta}(0)\rangle = N!\Delta^N(x)[\tfrac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma})b_N - \eta_{\alpha\beta}\eta_{\gamma\delta}c_N], \tag{1.4}$$

is completely determined by the coefficients a_N in the relation

$$\langle \mathrm{Tr}\,\phi^N(x), \mathrm{Tr}\,\phi^N(0)\rangle = \nu N! a_N \Delta^N(x), \tag{1.5}$$

where ν is the dimension of the matrix field.

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This can readily be seen from the pair of recurrence relations where $\nu = 4$:

$$
\begin{aligned}
9b_N &= 4a_{N+1} - (1-4c)a_N \\
9c_N &= a_{N+1} - \tfrac{1}{2}(5-4c)a_N.
\end{aligned}
\tag{1.6}
$$

For further use the result for (1.1) is quoted below:

$$
\begin{aligned}
&\langle [\exp \kappa\phi(x)]_{\alpha\beta}, [\exp \kappa\phi(0)]_{\gamma\delta}\rangle \\
&\quad = \frac{2}{9}\left(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\frac{\mathrm{d}}{\mathrm{d}(\kappa^2\Delta)}\right. \\
&\qquad \left. + \frac{4c-1}{18}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}) + \frac{5-2c}{18}\eta_{\alpha\beta}\eta_{\gamma\delta}\right) a(\Delta),
\end{aligned}
\tag{1.7}
$$

where

$$
\begin{aligned}
a(\Delta) = {} & (2 - 3z + z^2/2)\exp(-2cz) + [(2+3z-z^2) \\
& + \tfrac{3}{2}\pi z(z+\tfrac{1}{2})\mathbf{L}_0(z) - \tfrac{1}{2}\pi z^2\mathbf{L}_1(z)]\exp[(1-2c)z]|_{z=\kappa^2\Delta/2}.
\end{aligned}
\tag{1.8}
$$

$\mathbf{L}_0$ and $\mathbf{L}_1$ are the modified Struve functions of zeroth and first order, respectively.

In this paper it will be demonstrated how the Ashmore transform, which is essentially based on Siegel's integral, helps for finding a closed set of higher order superpropagators in closed form. The particular superpropagator

$$
\mathcal{T} = \langle [\exp \kappa\phi(x)]_{\alpha\beta}, [\exp \kappa\phi(0)]_{\gamma\delta}[\exp \kappa\phi(0)]_{\mu\nu}\rangle \tag{1.9}
$$

is calculated in detail as the result may serve to find the realistic gravity superpropagator

$$
\langle |-g(x)|^{2\omega} g_{\alpha\beta}(x) g_{\gamma\delta}(x), |-g(0)|^{2\omega} g_{\mu\nu}(0) g_{\rho\sigma}(0)\rangle.
$$

As a by-product of our main results, it is found that the asymptotic behaviour ($\Delta(x) \to \infty$) of a set of multi-matrix superpropagators can be anticipated; for any propagator of the set, a suitable gauge can be chosen which makes the propagator behave asymptotically well in the sense that no ambiguities arise in the Green function.

The importance of Siegel's integral in the present context is that it gives an expression for the determinant of a matrix field to an arbitrary power:

$$
\int_{\mathcal{D}} \mathrm{d}X |X|^{\mu}\, \mathrm{e}^{-\mathrm{Tr}(XY)} = \pi^{\nu(\nu-1)/4}\Gamma_\nu(\mu)|Y|^{-\mu-(\nu+1)/2}, \tag{1.10}
$$

where

$$
\begin{aligned}
&\Gamma_\nu(\mu) \equiv \Gamma(\mu+1)\Gamma(\mu+3/2)\Gamma(\mu+2)\ldots\Gamma[\mu+(\nu+1)/2], \\
&\mathrm{d}X = \mathrm{d}^{\nu(\nu+1)/2}x \equiv \prod_{\alpha\leq\beta} \mathrm{d}x_{\alpha\beta}.
\end{aligned}
\tag{1.11}
$$

The integration is taken over all $\frac{1}{2}\nu(\nu+1)$ elements $x_{\alpha\beta}$ which keep the $\nu \times \nu$ matrix X symmetric and positive definite. Initially representation (1.10) is defined for $\mathrm{Re}\,\mu > -1$. When $\nu = 1$, Siegel's integral reduces to the conventional definition of the gamma function (Von Siegel 1934).

2. The superpropagator $\mathcal{T}$

Due to Wick's reduction theorem we only need to consider the superpropagator

$$\langle \phi^L_{\alpha\beta}(x), \phi^M_{\gamma\delta}(0)\phi^N_{\mu\nu}(0)\rangle$$

with

$$L = M + N \tag{2.1}$$

to provide us with an answer for (1.9). Respecting the symmetry of the matrix field $\phi_{\alpha\beta}$ the general form of the expression (2.1) for arbitrary M and N reads:

$$\begin{aligned}\langle \phi^L_{\alpha\beta}(x), &\phi^M_{\gamma\delta}(0)\phi^N_{\mu\nu}(0)\rangle\\
&= L!\{A(L;M,N)\eta_{\alpha\beta}\eta_{\gamma\delta}\eta_{\mu\nu} + B(L;M,N)\eta_{\alpha\beta}(\eta_{\gamma\mu}\eta_{\delta\nu} + \eta_{\gamma\nu}\eta_{\delta\mu})\\
&\quad + C(L;M,N)\eta_{\gamma\delta}(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu}) + D(L;M,N)\eta_{\mu\nu}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma})\\
&\quad + E(L;M,N)[\eta_{\alpha\gamma}(\eta_{\beta\mu}\eta_{\delta\nu} + \eta_{\beta\nu}\eta_{\delta\mu}) + \eta_{\alpha\delta}(\eta_{\beta\mu}\eta_{\gamma\nu} + \eta_{\beta\nu}\eta_{\gamma\mu})\\
&\quad + \eta_{\beta\gamma}(\eta_{\alpha\mu}\eta_{\delta\nu} + \eta_{\alpha\nu}\eta_{\delta\mu}) + \eta_{\beta\delta}(\eta_{\alpha\mu}\eta_{\gamma\nu} + \eta_{\alpha\nu}\eta_{\beta\mu})]\}\Delta^L(x),\end{aligned} \tag{2.2}$$

with $A(L;M,N),\ldots,E(L;M,N)$ coefficients only depending by (2.1) on M and N.

We will now prove that once the coefficients b_N and c_N in (1.6) have been spelled out explicitly, it is sufficient to determine $U(L;M,N)$ in

$$\langle \operatorname{Tr}\phi^L(x), \operatorname{Tr}\phi^M(0)\operatorname{Tr}\phi^N(0)\rangle = \nu L!\,U(L;M,N)\Delta^L(x), \tag{2.3}$$

for all possible values of L, M and N, in order to evaluate $A(L;M,N),\ldots,E(L;M,N)$.

Taking the traces in (2.2), comparison with (2.3) yields:

$$\begin{aligned}\nu^2 A(L;M,N) &+ 2\nu[B(L;M,N) + C(L;M,N) + D(L;M,N)] + 8E(L;M,N)\\
&= U(L;M,N),\end{aligned} \tag{2.4}$$

while setting δ equal to μ in (2.2) and summing over μ gives, with the help of equation (1.4), the two relations

$$\begin{aligned}&A(L;M,N) + (\nu+1)B(L;M,N) + 2E(L;M,N) = -c_L\\
&C(L;M,N) + D(L;M,N) + (\nu+2)E(L;M,N) = \tfrac{1}{2}b_L.\end{aligned} \tag{2.5}$$

Finally, direct Wick expansion on (2.3), taking (1.3) into consideration, gives:

$$\begin{aligned}\langle \operatorname{Tr}\phi^L(x), &\operatorname{Tr}\phi^M(0)\operatorname{Tr}\phi^N(0)\rangle\\
&= \{M\langle \phi^{L-1}_{\sigma\tau}, \phi^{M-1}_{\sigma\tau}\operatorname{Tr}\phi^N\rangle + N\langle \phi^{L-1}_{\sigma\tau}, \operatorname{Tr}\phi^M\phi^{N-1}_{\sigma\tau}\rangle\\
&\quad - cM\langle \operatorname{Tr}\phi^{L-1}, \operatorname{Tr}\phi^{M-1}\operatorname{Tr}\phi^N\rangle - cN\langle \operatorname{Tr}\phi^{L-1}, \operatorname{Tr}\phi^M\operatorname{Tr}\phi^{N-1}\rangle\}\Delta(x)\end{aligned} \tag{2.6}$$

$$= L\{\langle \phi^{L-1}_{\sigma\tau}, \phi^{M-1}_{\sigma\tau}\operatorname{Tr}\phi^N\rangle - c\langle \operatorname{Tr}\phi^{L-1}, \operatorname{Tr}\phi^{M-1}\operatorname{Tr}\phi^N\rangle\}\Delta(x) \tag{2.7}$$

$$= L\{\langle \phi^{L-1}_{\sigma\tau}, \operatorname{Tr}\phi^M\phi^{N-1}_{\sigma\tau}\rangle - c\langle \operatorname{Tr}\phi^{L-1}, \operatorname{Tr}\phi^M\operatorname{Tr}\phi^{N-1}\rangle\}\Delta(x). \tag{2.8}$$

Using equations (2.2) and (2.3), the relations (2.7) and (2.8) can be transformed to give the following relations between $A(L; M, N), \ldots, E(L; M, N)$ and $U(L; M, N)$:

$$\begin{aligned} U(L+1; M+1, N)+cU(L; M, N) \\ &= \nu A(L; M, N)+2[B(L; M, N)+C(L; M, N)] \\ &\quad +\nu(\nu+1)D(L; M, N)+4(\nu+1)E(L; M, N) \end{aligned} \tag{2.9}$$

$$\begin{aligned} U(L+1; M, N+1)+cU(L; M, N) \\ &= \nu A(L; M, N)+2[B(L; M, N)+D(L; M, N)] \\ &\quad +\nu(\nu+1)C(L; M, N)+4(\nu+1)E(L; M, N). \end{aligned} \tag{2.10}$$

We further introduce two coefficients $S(L; M, N)$ and $T(L; M, N)$ defined by

$$S(L; M, N) = U(L+1; M+1, N) - U(L+1; M, N+1) \tag{2.11}$$

$$T(L; M, N) = U(L+1; M+1, N) + U(L+1; M, N+1) + 2cU(L; M, N). \tag{2.12}$$

The set of independent equations (2.4), (2.5), (2.9) and (2.10) satisfied by the coefficients $A(L; M, N), \ldots, E(L; M, N)$ yields a unique solution for these coefficients which reads, in the case $\nu = 4$:

$$\begin{aligned} 18A(L; M, N) &= \tfrac{13}{8}U(L; M, N) - \tfrac{3}{4}T(L; M, N) + \tfrac{7}{4}b_L + 2c_L \\ 18B(L; M, N) &= -\tfrac{3}{8}U(L; M, N) + \tfrac{1}{4}T(L; M, N) - \tfrac{5}{4}b_L - 4c_L \\ 18C(L; M, N) &= -\tfrac{3}{8}U(L; M, N) + \tfrac{3}{4}T(L; M, N) - \tfrac{1}{2}S(L; M, N) - \tfrac{9}{4}b_L \\ 18D(L; M, N) &= -\tfrac{3}{8}U(L; M, N) + \tfrac{3}{4}T(L; M, N) + \tfrac{1}{2}S(L; M, N) - \tfrac{9}{4}b_L \\ 18E(L; M, N) &= \tfrac{1}{8}U(L; M, N) - \tfrac{1}{4}T(L; M, N) + \tfrac{9}{4}b_L. \end{aligned} \tag{2.13}$$

$A(L; M, N), \ldots, E(L; M, N)$ only depend on b_L, c_L and U coefficients which proves our statement.

It is stated without proof that every superpropagator of the form

$$\langle \phi^L_{\alpha\beta}(x), \phi^{M_1}_{\alpha_1\beta_1}(0)\phi^{M_2}_{\alpha_2\beta_2}(0) \ldots \phi^{M_k}_{\alpha_k\beta_k}(0)\rangle,$$

with

$$\sum_{i=1}^{k} M_i = L, \qquad k \in \mathbb{N}, \qquad k \geqslant 2, \tag{2.14}$$

can be found from the knowledge of a similar superpropagator with k decreased by one unit, and from the knowledge of the superpropagator

$$\langle \operatorname{Tr} \phi^L(x), \operatorname{Tr} \phi^{M_1}(0) \operatorname{Tr} \phi^{M_2}(0) \ldots \operatorname{Tr} \phi^{M_k}(0)\rangle. \tag{2.15}$$

We note that with increasing value of k, the number S_k of different terms in the general form of the superpropagator grows very rapidly and is given by

$$S_k = (2k+2)!/(k+1)!2^{k+1}.$$

In the case $k = 3$ there are 105 different terms which from symmetry requirements must be ordered into 17 groups, hence requiring 17 coefficients.

We finally note that equation (2.6) has not been used to find solution (2.13), but it is readily seen that equation (2.6) is not independent from equations (2.7) and (2.8).

3. An integral representation

Defining an expectation value $\mathcal{R}$ by:

$$\mathcal{R}=\langle|1+\kappa\phi(x)|^{-\mu-(\nu+1)/2},|1+\kappa'\phi(0)|^{-\mu'-(\nu+1)/2}|1+\kappa''\phi(0)|^{-\mu''-(\nu+1)/2}\rangle \tag{3.1}$$

Siegel's integral (1.10) is applied three times to $\mathcal{R}$ with the substitutions $|Y|=|1+\kappa\phi|$, $|Y'|=|1+\kappa'\phi|$, and $|Y''|=|1+\kappa''\phi|$. With the use of the formula

$$\begin{aligned}&\langle\exp(-\mathrm{Tr}\,A\phi(x)),\exp(-\mathrm{Tr}\,B\phi(0))\exp(-\mathrm{Tr}\,C\phi(0))\rangle\\&\qquad=\exp[(\mathrm{Tr}\,AB-c\,\mathrm{Tr}\,A\,\mathrm{Tr}\,B)\Delta(x)]\exp[(\mathrm{Tr}\,AC-c\,\mathrm{Tr}\,A\,\mathrm{Tr}\,C)\Delta(x)],\end{aligned} \tag{3.2}$$

which holds for arbitrary $(\nu\times\nu)$ matrices, A, B and C, we get for $\mathcal{R}$:

$$\begin{aligned}\mathcal{R}=&\int_{\mathcal{D}}\int_{\mathcal{D}}\int_{\mathcal{D}}\frac{\mathrm{d}X\,\mathrm{d}X'\,\mathrm{d}X''|X|^{\mu}|X'|^{\mu'}|X''|^{\mu''}}{\pi^{3\nu(\nu-1)/4}\Gamma_\nu(\mu)\Gamma_\nu(\mu')\Gamma_\nu(\mu'')}\langle\exp[-\mathrm{Tr}X(1+\kappa\phi(x))],\\&\exp[-\mathrm{Tr}\,X'(1+\kappa'\phi(0))]\exp[-\mathrm{Tr}\,X''(1+\kappa''\phi(0))]\rangle\\=&\int_{\mathcal{D}}\int_{\mathcal{D}}\int_{\mathcal{D}}\frac{\mathrm{d}X\,\mathrm{d}X'\,\mathrm{d}X''|X|^{\mu}|X'|^{\mu'}|X''|^{\mu''}}{\pi^{3\nu(\nu-1)/4}\Gamma_\nu(\mu)\Gamma_\nu(\mu')\Gamma_\nu(\mu'')}\exp[-(\mathrm{Tr}\,X+\mathrm{Tr}\,X'+\mathrm{Tr}\,X'')]\\&\times\exp[\kappa\kappa'(\mathrm{Tr}\,XX'-c\,\mathrm{Tr}\,X\,\mathrm{Tr}\,X')\Delta(x)]\\&\times\exp[\kappa\kappa''(\mathrm{Tr}\,XX''-c\,\mathrm{Tr}\,X\,\mathrm{Tr}\,X'')\Delta(x)]\\=&\int_{\mathcal{D}}\frac{\mathrm{d}X|X|^{\mu}}{\pi^{\nu(\nu-1)/4}\Gamma_\nu(\mu)}[\exp(-\mathrm{Tr}\,X)]|1-\kappa\kappa'(X-c\,\mathrm{Tr}\,X)\Delta(x)|^{-\mu'-(\nu+1)/2}\\&\times|1-\kappa\kappa''(X-c\,\mathrm{Tr}\,X)\Delta(x)|^{-\mu''-(\nu+1)/2},\end{aligned} \tag{3.3}$$

where in the last transition Siegel's integral has been applied twice in reversed order. Differentiating successively with respect to μ, μ' and μ'' on both sides of (3.3) at the point $-(\nu+1)/2$, one gets

$$\begin{aligned}&\langle-\mathrm{Tr}\ln(1+\kappa\phi(x)),[-\mathrm{Tr}\ln(1+\kappa'\phi(0))][-\mathrm{Tr}\ln(1+\kappa''\phi(0))]\rangle\\&\quad=\partial_\mu|_{\mu=-(\nu+1)/2}\int_{\mathcal{D}}\frac{\mathrm{d}X|X|^{\mu}}{\pi^{\nu(\nu-1)/4}\Gamma_\nu(\mu)}[\exp(-\mathrm{Tr}\,X)]\{-\mathrm{Tr}\ln[1-\kappa\kappa'(X-c\,\mathrm{Tr}\,X)\\&\qquad\times\Delta(x)]\}\{-\mathrm{Tr}\ln[1-\kappa\kappa''(X-c\,\mathrm{Tr}\,X)\Delta(x)]\}.\end{aligned} \tag{3.4}$$

Terms of order $(\kappa\kappa')^M(\kappa\kappa'')^N=\kappa^L\kappa'^M\kappa''^N$ are taken from both sides of (3.4) to reach the integral representation

$$\langle\mathrm{Tr}\,\phi^L(x),\mathrm{Tr}\,\phi^M(0)\,\mathrm{Tr}\,\phi^N(0)\rangle=L\Delta^L(x)\,\partial_\mu|_{\mu=-(\nu+1)/2}I_\nu^{(c)}(\mu;M,N), \tag{3.5}$$

$$I_\nu^{(c)}(\mu;M,N)=\int_{\mathcal{D}}\frac{\mathrm{d}X|X|^{\mu}}{\pi^{\nu(\nu-1)/4}\Gamma_\nu(\mu)}[\exp(-\mathrm{Tr}\,X)]\,\mathrm{Tr}[(X-c\,\mathrm{Tr}\,X)^M]\,\mathrm{Tr}[(X-c\,\mathrm{Tr}\,X)^N]. \tag{3.6}$$

We note that setting either M or N equal to zero in (3.5) and (3.6) with the definition $\mathrm{Tr}\,\phi^0(x)=\nu$, gives the Ashmore result. To simplify, a relation is sought between

$I_\nu^{(c)}(\mu; M, N)$ and $I_\nu^{(0)}(\mu; M, N)$:

$$I_\nu^{(c)}(\mu; M, N) = \sum_{m=0}^{M} \sum_{n=0}^{N} \binom{M}{n}\binom{N}{n} \int_{\mathscr{D}} \frac{\mathrm{d}X|X|^\mu}{\pi^{\nu(\nu-1)/4}\Gamma_\nu(\mu)}[\exp(-\mathrm{Tr}\, X)]$$
$$\times(-c\, \mathrm{Tr}\, X)^{M+N-m-n}\, \mathrm{Tr}(X^m)\, \mathrm{Tr}(X^n)$$
$$= \sum_{m=0}^{M} \sum_{n=0}^{N} \binom{M}{m}\binom{N}{n}(c\partial_b)^{M+N-m-n} \int_{\mathscr{D}} \frac{\mathrm{d}X|X|^\mu}{\pi^{\nu(\nu-1)/4}\Gamma_\nu(\mu)}$$
$$\times[\exp(-b\, \mathrm{Tr}\, X)]\, \mathrm{Tr}(X^m)\mathrm{Tr}\,(X^n)|_{b=1}.$$

Rescaling X by the factor b results in:

$$I_\nu^{(0)}(\mu; M, N) = \sum_{m=0}^{M} \sum_{n=0}^{N} \binom{M}{m}\binom{N}{n}$$
$$\times(-c)^{M+N-m-n}\frac{\Gamma(M+N+\mu\nu+\nu(\nu+1)/2)}{\Gamma(m+n+\mu\nu+\nu(\nu+1)/2)} I_\nu^{(0)}(\mu; m, n). \tag{3.7}$$

The arguments outlined in this section could be repeated for all propagators (2.15). For arbitrary values of k the problem is always reduced to the calculation of the multiple integral

$$I_\nu^{(0)}(\mu; m_1, m_2, \ldots, m_k)$$
$$= \int_{\mathscr{D}} \frac{\mathrm{d}X|X|^\mu}{\pi^{\nu(\nu-1)/4}\Gamma_\nu(\mu)}[\exp(-\mathrm{Tr}\, X)]\, \mathrm{Tr}(X^{m_1})\, \mathrm{Tr}(X^{m_2}) \ldots \mathrm{Tr}(X^{m_k}). \tag{3.8}$$

The multiple integrals (3.8) for arbitrary $k>0$ have the common feature of being involved only in the integrand functions of X invariant under similarity transformations. This property enables one to bring the symmetrical X into diagonal form. Although with increasing k the algebraic calculations become more tedious, a standard technique can be outlined to evaluate (3.8). In the next sections $I_4^{(0)}(\mu; m, n)$ will be calculated for arbitrary m and n. Note that it is only for simplicity that $\nu=4$ since the calculations may be carried out for other integer values of ν as well.

4. A formula for *U(L; M, N)*

As stated at the end of the previous section, the matrix X in

$$I_4^{(0)}(\mu; m, n) = \frac{1}{\pi^3}\int_{\mathscr{D}} \frac{\mathrm{d}X|X|^\mu}{\Gamma_4(\mu)}[\exp(-\mathrm{Tr}\, X)]\, \mathrm{Tr}(X^m)\, \mathrm{Tr}(X^n), \tag{4.1}$$

is diagonalized to Λ by

$$X = S(\theta)\Lambda\tilde{S}(\theta),$$

where θ symbolizes a set of $\nu(\nu-1)/2$ angular parameters. Taking into account the Jacobian of the transformation defined by

$$\mathrm{d}^{\nu(\nu+1)/2}x = \prod_{j>i}|\lambda_j - \lambda_i| \prod_k \mathrm{d}\lambda_k J(\theta)\, \mathrm{d}^{\nu(\nu-1)/2}\theta,$$

where J is depending solely on the angular parameters comprised in (4.1), becomes

$$I_4^{(0)}(\mu; m, n) = \frac{\gamma_4}{\Gamma_4(\mu)} \prod_{k=1}^{4} \int_0^\infty d\lambda_k\, e^{-\lambda_k} \lambda_k^\mu \prod_{j>i} |\lambda_j - \lambda_i| \sum_{l=1}^{4} \lambda_l^m \sum_{p=1}^{4} \lambda_p^n, \tag{4.2}$$

with γ_4 a normalization constant determined by the condition that $I_4^{(0)}(\mu; 0, 0) = \nu^2 = 16$.

Now, the Ashmore algorithm can be applied to the remaining fourfold integral (4.2). The result is:

$$I_4^{(0)}(\mu; m, n) = \frac{4!\gamma_4}{\Gamma_4(\mu)} \sum_{l=1}^{4} \sum_{p=1}^{4} (\text{Pfaff})_{lp}, \tag{4.3}$$

where $(\text{Pfaff})_{lp}$ is the Pfaffian which is deduced from the scheme

$$\begin{vmatrix} a_{1,2} & a_{1,3} & a_{1,4} \\ & a_{2,3} & a_{2,4} \\ & & a_{3,4} \end{vmatrix}, \tag{4.4}$$

by adding m to every subscript which is equal to l and also adding n to every subscript which is equal to p, and every a symbol obtained in this way is given by

$$a_{ij} = \int_0^\infty d\lambda \int_0^\infty d\lambda'\, E_i(\lambda) E_j(\lambda')\, \text{sgn}(\lambda' - \lambda), \tag{4.5}$$

with

$$E_n(\lambda) = \lambda^{\mu+n-1} e^{-\lambda}.$$

As the dimension equals 4, equation (4.3) can be rewritten as:

$$I_4^{(0)}(\mu; m, n) = \frac{4!\gamma_4}{\Gamma_4(\mu)} 4 \sum_{i=1}^{3} \sum_{j=i+1}^{4} (-1)^P (a_{ij}^{(m,n)} a_{i'j'}^{(0)} + a_{ij}^{(m)} a_{i'j'}^{(n)}), \tag{4.6}$$

where the indices i' and j' are the indices of the cofactor of the element a_{ij} in the scheme (4.4) and $(-1)^P$ is the sign of the permutation $(iji'j')$ of the natural order (1234). The quantities $a_{ij}^{(m,n)}$ and $a_{ij}^{(m)}$ are defined by

$$a_{ij}^{(m,n)} = \tfrac{1}{4}(a_{i+m+n,j} + a_{i,j+m+n} + a_{i+m,j+n} + a_{i+n,j+m}), \tag{4.7}$$

$$a_{ij}^{(m)} = \tfrac{1}{2}(a_{i,j+m} + a_{i+m,j}). \tag{4.8}$$

Substituting (4.5) in the right-hand side of (4.8) it is found that

$$a_{ij}^{(m)} = (\tfrac{1}{2})^{2\mu+i+j+m} \Gamma(2\mu + i + j + m) \alpha_{ij}^{(m)}, \tag{4.9}$$

with

$$\begin{aligned} \alpha_{ij}^{(m)} &= (1+\partial)^m \partial_{ij} \alpha(z)|_{z=0} \equiv \partial^m e^z \partial_{ij} \alpha(z)|_{z=0} \\ \partial &\equiv \partial/\partial z \\ \partial_{ij} &\equiv (1-\partial)^{i-1}(1+\partial)^{j-1} - (1-\partial)^{j-1}(1+\partial)^{i-1} \\ \alpha(z) &= \frac{\sqrt{\pi}\Gamma(\mu+1)}{(z/2)^{\mu+1/2}} \mathbf{L}_{\mu+1/2}(z) = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(k+\mu+2)\Gamma(k+3/2)} \left(\frac{z}{2}\right)^{2k+1} \end{aligned} \tag{4.10}$$

where $\mathbf{L}$ means a modified Struve function.

With the same notations it is found from the substitution of (4.5) in the right-hand side of (4.7) that

$$a_{ij}^{(m,n)} = (\tfrac{1}{2})^{2\mu+i+j+m+n}\Gamma(2\mu+i+j+m+n)\alpha_{ij}^{(m,n)}, \tag{4.11}$$

with

$$\alpha_{ij}^{(m,n)} = \tfrac{1}{2}\alpha_{ij}^{(m+n)} + \tfrac{1}{4}\alpha_{i+m,j+n}^{(0)} + \tfrac{1}{4}\alpha_{i+n,j+m}^{(0)}. \tag{4.12}$$

Recalling the definition (1.11) of $\Gamma_\nu(\mu)$ it is easy to show that

$$\Gamma_4(\mu) = \pi 2^{-4\mu-4}\Gamma(2\mu+2)\Gamma(2\mu+4). \tag{4.13}$$

Finally it is noted that

$$i'+j' = 10-i-j. \tag{4.14}$$

From the comparison of the equations (2.3) and (3.5) one finds, having used the results (3.7) and (4.6)–(4.14), the following expression for the coefficients $U(L; M, N)$:

$$\begin{aligned} U(L; M, N) = {} & \frac{4!\gamma_4}{\pi(L-1)!}\left(\frac{1}{2}\right)^6 \sum_{m=0}^{M}\sum_{n=0}^{N}\left(\frac{1}{2}\right)^{m+n}\binom{M}{m}\binom{N}{n}(-c)^{M+N-m-n} \\ & \times \sum_{i=1}^{3}\sum_{j=i+1}^{4} \partial_\mu|_{-5/2}\left(\frac{\Gamma(M+N+4\mu+10)}{\Gamma(m+n+4\mu+10)\Gamma(2\mu+2)\Gamma(2\mu+4)}\right. \\ & \times[\Gamma(2\mu+i+j+m+n)\Gamma(2\mu+10-i-j)(-1)^p\alpha_{ij}^{(m,n)}\alpha_{i'j'}^{(0)} \\ & \left.+\Gamma(2\mu+i+j+m)\Gamma(2\mu+10-i-j+n)(-1)^p\alpha_{ij}^{(m)}\alpha_{i'j'}^{(n)}]\right). \end{aligned} \tag{4.15}$$

Apart from the factor $\Gamma(M+N+4\mu+10)$, the expression in the large parentheses is identically zero at $\mu = -5/2$.

Indeed as $3 \leqslant i+j \leqslant 7$, the two Γ functions in the numerator only have a pole simultaneously for some i and j value when $m = n = 0$. In that case however the three Γ functions in the denominator have a pole and the result is zero. If $m+n \neq 0$, the result remains zero as the denominator has a pole of at least one order greater than the order of the pole in the numerator. Consequently (4.15) can be rewritten as

$$\begin{aligned} U(L; M, N) = {} & \frac{4!\gamma_4}{\pi}\left(\frac{1}{2}\right)^6 \sum_{m=0}^{M}\sum_{n=0}^{N}\left(\frac{1}{2}\right)^{m+n}\binom{M}{m}\binom{N}{n}(-c)^{M+N-m-n} \\ & \times \sum_{i=1}^{3}\sum_{j=i+1}^{4} \partial_\mu|_{-5/2}\left(\frac{\Gamma(2\mu+i+j+m+n)\Gamma(2\mu+10-i-j)}{\Gamma(2\mu+2)\Gamma(2\mu+4)\Gamma(m+n+4\mu+10)}(-1)^p\alpha_{ij}^{(m,n)}\alpha_{i'j'}^{(0)}\right. \\ & \left.+\frac{\Gamma(2\mu+i+j+m)\Gamma(2\mu+10-i-j+n)}{\Gamma(2\mu+2)\Gamma(2\mu+4)\Gamma(m+n+4\mu+10)}(-1)^p\alpha_{ij}^{(m)}\alpha_{i'j'}^{(n)}\right), \end{aligned} \tag{4.16}$$

and this expression is valid for all values of $L \geqslant 0$.

5. Reduction of *U(L; M, N)*

We shall now further reduce the right-hand side of (4.16). As analogous argument to the one at the end of the previous section shows that, given fixed values for m and n, only those values of i and j must be taken into account which make the argument of at least one of the two Γ functions in the numerator become zero or a negative integer. Let us first calculate

$$\mathscr{P}=\left(\frac{1}{2}\right)^{m+n}\sum_{i=1}^{3}\sum_{j=i+1}^{4}\partial_\mu|_{-5/2}\frac{\Gamma(2\mu+i+j+m+n)\Gamma(2\mu+10-i-j)}{\Gamma(2\mu+2)\Gamma(2\mu+4)\Gamma(m+n+4\mu+10)}(-1)^p\alpha_{ij}^{(m,n)}\alpha_{i'j'}^{(0)}. \tag{5.1}$$

If $m+n=0$, a first non-zero contribution to $\mathscr{P}$ is found for $i+j=5$. With the intermediate result

$$\partial_\mu|_{-5/2}\left(\frac{\Gamma^2(2\mu+5)}{\Gamma(2\mu+2)\Gamma(2\mu+4)\Gamma(4\mu+10)}\right)=4\left(\frac{(2\mu+2)(2\mu+3)(2\mu+4)^2}{\Gamma(4\mu+11)}\right)_{\mu=-5/2}=24,$$

the contribution to $\mathscr{P}$ becomes

$$24\delta_{m+n,0}(\alpha_{14}^{(0,0)}\alpha_{23}^{(0)}+\alpha_{23}^{(0,0)}\alpha_{14}^{(0)}).$$

Proceeding in a similar way for all m and n values the following expression for $\mathscr{P}$ is found:

$$\begin{aligned}\mathscr{P}=(\tfrac{1}{2})^{m+n}\{&6[(m+n)(m+n+1)\alpha_{34}^{(m,n)}\alpha_{12}^{(0)}+2(m+n)\alpha_{24}^{(m,n)}\alpha_{13}^{(0)}\\&+2\alpha_{14}^{(m,n)}\alpha_{23}^{(0)}+2\alpha_{23}^{(m,n)}\alpha_{14}^{(0)}]+12\delta_{m+n,2}\alpha_{12}^{(m,n)}\alpha_{34}^{(0)}\\&-12\delta_{m+n,1}(\alpha_{13}^{(m,n)}\alpha_{24}^{(0)}+\alpha_{12}^{(m,n)}\alpha_{34}^{(0)})+12\delta_{m+n,0}(\alpha_{14}^{(0,0)}\alpha_{23}^{(0)}+\alpha_{23}^{(0,0)}\alpha_{14}^{(0)})\}.\end{aligned} \tag{5.2}$$

A second contribution to $U(L; M, N)$ comes from

$$\mathscr{Q}=\left(\frac{1}{2}\right)^{m+n}\sum_{i=1}^{3}\sum_{j=i+1}^{4}\partial_\mu|_{-5/2}\frac{\Gamma(2\mu+i+j+m)\Gamma(2\mu+10-i-j+n)}{\Gamma(2\mu+2)\Gamma(2\mu+4)\Gamma(m+n+4\mu+10)}(-1)^p\alpha_{ij}^{(m)}\alpha_{i'j'}^{(n)}. \tag{5.3}$$

A similar calculation to the one used for $\mathscr{P}$ gives

$$\begin{aligned}\mathscr{Q}=(\tfrac{1}{2})^{m+n}\{&[12\delta_{m2}\alpha_{12}^{(2)}\alpha_{34}^{(n)}-12\delta_{m1}((n+1)\alpha_{12}^{(1)}\alpha_{34}^{(n)}+\alpha_{13}^{(1)}\alpha_{24}^{(n)})+6\delta_{m0}(n(n+1)\alpha_{12}^{(0)}\alpha_{34}^{(n)}\\&+2n\alpha_{13}^{(0)}\alpha_{24}^{(n)}+2\alpha_{23}^{(0)}\alpha_{14}^{(n)}+2\alpha_{14}^{(0)}\alpha_{23}^{n})]+(m\leftrightarrow n)\}.\end{aligned} \tag{5.4}$$

The normalization constant γ_4 is fixed by computing the coefficient $U(0; 0, 0)$. For this purpose, use is made of some values for $\alpha_{ij}^{(q)}$ $(q\leqslant 2)$ which can be read off from table 1,

Table 1. Some values of $\alpha_{ij}^{(q)}$ and $\alpha_{ij}^{(q,\tau)}$ $(i>j,\ q\leqslant 2,\ q+\tau\leqslant 2)$.

Superscript	α_{12}	α_{13}	α_{14}	α_{23}	α_{24}	α_{34}
(0), (1) (0, 0), (0, 1), (1, 0) (1, 1)	$-\frac{4}{3}$	$-\frac{8}{3}$	$-\frac{4}{3}$	-4	-8	$+4$
(2) (0, 2), (2, 0)	$+\frac{4}{3}$	$+\frac{8}{3}$	$+\frac{52}{3}$	-12	-24	$+\frac{20}{3}$

and we find that

$$U(0;0,0)=\frac{4!\gamma_4}{\pi 2^6}96\alpha_{14}^{(0)}\alpha_{23}^{(0)}=\frac{192}{\pi}\gamma_4$$

and γ_4 must be equal to $\pi/12$ in order to give the exact result $U(0;0,0)=16$, which can be deduced from equation (2.3) with $\nu=4$. In a similar way it is calculated from equations (4.6), (5.1)–(5.4) that

$$U(1;0,1)=U(1;1,0)=4(1-4c),$$
$$U(2;1,1)=(1-4c)^2,$$

which is in perfect agreement with the value of the same coefficients calculated from equations (1.3) and (2.3) by direct Wick reduction. Substituting the expressions (5.2) and (5.4) in equation (4.16) brings $U(L;M,N)$ firstly into the form:

$$\begin{aligned}U(L;M,N)=&4(-c)^{M+N}-3(-c)^{M+N-1}(M+N)+\tfrac{1}{4}(-c)^{M+N-2}[(M-N)^2-(M+N)]\\&-\frac{1}{4}\sum_{m=0}^{M}\sum_{n=0}^{N}\binom{N}{n}\binom{M}{m}(-c)^{M+N-m-n}\left(\frac{1}{2}\right)^{m+n}[(m+n)(m+n+1)\alpha_{34}^{(m,n)}\\&+4(m+n)\alpha_{24}^{(m,n)}+6\alpha_{14}^{(m,n)}+2\alpha_{23}^{(m,n)}]\\&+\frac{1}{16}M(M-1)(-c)^{M-2}\sum_{n=0}^{N}\binom{N}{n}(-c)^{N-n}\left(\frac{1}{2}\right)^{n}\alpha_{34}^{(n)}\\&+\frac{1}{16}N(N-1)(-c)^{N-2}\sum_{m=0}^{M}\binom{M}{m}(-c)^{M-m}\left(\frac{1}{2}\right)^{m}\alpha_{34}^{(m)}\\&+\frac{1}{4}N(-c)^{N-1}\sum_{m=0}^{M}\binom{M}{m}(-c)^{M-m}\left(\frac{1}{2}\right)^{m}[(m+1)\alpha_{34}^{(m)}+2\alpha_{24}^{(m)}]\\&+\frac{1}{4}M(-c)^{M-1}\sum_{n=0}^{N}\binom{N}{n}(-c)^{N-n}\left(\frac{1}{2}\right)^{n}[(n+1)\alpha_{34}^{(n)}+2\alpha_{24}^{(n)}]\\&-\frac{1}{4}(-c)^{N}\sum_{m=0}^{M}\binom{M}{m}\\&\times(-c)^{M-m}\left(\frac{1}{2}\right)^{m}[m(m+1)\alpha_{34}^{(m)}+4m\alpha_{24}^{(m)}+6\alpha_{14}^{(m)}+2\alpha_{23}^{(m)}]\\&-\frac{1}{4}(-c)^{M}\sum_{n=0}^{N}\binom{N}{n}(-c)^{N-n}\left(\frac{1}{2}\right)^{n}[n(n+1)\alpha_{34}^{(n)}+4n\alpha_{24}^{(n)}+6\alpha_{14}^{(n)}+2\alpha_{23}^{(n)}].\end{aligned}\tag{5.5}$$

To obtain the result (5.5) again, some values from table 1 have been used. Introducing the definitions (4.10) into (5.5), the following expression for $U(L;M,N)$ is found:

$$\begin{aligned}U(L;M,N)=&4(-c)^{M+N}-3(-c)^{M+N-1}(M+N)+\tfrac{1}{4}(-c)^{M+N-2}[(M-N)^2-(M+N)]\\&-\frac{1}{4}\sum_{m=0}^{M}\sum_{n=0}^{N}\binom{N}{n}\binom{M}{m}(-c)^{M+N-m-n}\left(\frac{1}{2}\right)^{m+n}\{\tfrac{1}{2}\partial^{m+n}+\tfrac{1}{4}[(2-\partial)^m\partial^n\\&+(2-\partial)^n\partial^m]\}\,e^z[2(m+n)(m+n-1)(1-\partial^2)^2+4(m+n)(1-\partial^2)(5-\partial^2)\end{aligned}$$

$$+8(5+\partial^2)]\partial\alpha(z)|_{z=0}+\left\{\left[\frac{1}{16}M(M-1)(-c)^{M-2}\sum_{n=0}^{N}\binom{N}{n}\right.\right.$$

$$\times(-c)^{N-n}\left(\frac{1}{2}\right)^n\partial^n\, e^z(1-\partial^2)^2 2\partial\alpha(z)|_{z=0}+\frac{1}{4}M(-c)^{M-1}$$

$$\times\sum_{n=0}^{N}\binom{N}{n}(-c)^{N-n}\left(\frac{1}{2}\right)^n\partial^n\, e^z[2n(1-\partial^2)^2+2(1-\partial^2)(5-\partial^2)]\partial\alpha(z)|_{z=0}$$

$$-\frac{1}{4}(-c)^M\sum_{n=0}^{N}\binom{N}{n}$$

$$\times(-c)^{N-n}\left(\frac{1}{2}\right)^n\partial^n\, e^z[2n(n-1)(1-\partial^2)^2+4n(1-\partial^2)(5-\partial^2)$$

$$\left.\left.+8(5+\partial^2)]\partial\alpha(z)|_{z=0}\right]+\{m\leftrightarrow n, M\leftrightarrow N\}\right\}. \tag{5.6}$$

The result (5.6) is essential for the calculation of the gravity superpropagator in the following section.

6. The gravity superpropagator

If one tries to calculate the superpropagator

$$\langle|-g(x)|^{\omega}g_{\alpha\beta}(x), |-g(0)|^{\omega}g_{\gamma\delta}(0)|-g(0)|^{\omega}g_{\mu\nu}(0)\rangle, \tag{6.1}$$

with the graviton field $h_{\mu\nu}(x)$ defined by the relation

$$g_{\mu\nu}(x)=[\exp \kappa h(x)]_{\mu\nu}, \tag{6.2}$$

and having a free propagator of the form

$$\langle h_{\alpha\beta}(x), h_{\gamma\delta}(0)\rangle=\tfrac{1}{2}[\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma}-2c'\eta_{\alpha\beta}\eta_{\gamma\delta}]\Delta(x), \tag{6.3}$$

it is sufficient to calculate the superpropagator $\mathcal{T}$ defined in (1.9), with the matrix field $\phi_{\alpha\beta}(x)$ defined in (1.2) and the free propagator (1.3). It was already demonstrated by Ashmore that the gauge parameters c and c' are related in the following manner:

$$(1-4c)=(1-4c')(1+4\omega)^2. \tag{6.4}$$

In what follows we shall calculate the superpropagator $\mathcal{T}$, keeping in mind that at the end, the parameter c must be expressed in terms of the gauge parameter c' and the weight parameter ω.

Expanding the right-hand side of formula (1.9) with respect to the parameter κ we find

$$\mathcal{T}=\sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\frac{(\kappa^2)^{M+N}}{(M+N)!M!N!}\langle\phi^L_{\alpha\beta}(x), \phi^M_{\gamma\delta}(0)\phi^N_{\mu\nu}(0)\rangle\delta_{L,M+N}. \tag{6.5}$$

Replacing the superpropagator in the right-hand side of (6.5) by its general form (2.2) and afterwards introducing the solution (2.13) in the expression obtained, it is immediately seen that only

$$F_{ex}=\sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\frac{\lambda^{M+N}}{(M+N)!M!N!}L!F(L;M,N)\delta_{L,M+N}, \tag{6.6}$$

with

$$\lambda = \kappa^2 \Delta(x), \tag{6.7}$$

and $F(L; M, N)$ standing for $U(L; M, N)$, $S(L; M, N)$, $T(L; M, N)$, b_L or c_L, have to be calculated.

It is easy to show with the help of the definition (2.11) and the expression (5.6), that $S_{ex} = 0$, which also means that $C_{ex} = D_{ex}$ according to (2.13). This result could have been predicted from the fact that the superpropagator (1.9) is invariant to an interchange of the (γ, δ) indices with the (μ, ν) indices.

Further, the formula

$$b_{ex} = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \frac{\lambda^{M+N}}{M!N!} b_{M+N} = \sum_{L=0}^{\infty} \frac{(2\lambda)^L}{L!} b_L,$$

and an analogous formula for c_{ex} show that b_{ex} and c_{ex} can be deduced from the Ashmore calculus and may be more precisely recovered from the formulae (1.7) and (1.8) after replacing κ^2 by $2\kappa^2$ in (1.8). The evaluation of T_{ex} is also much simplified by noting that the identity

$$\partial_\lambda \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \frac{\lambda^{M+N}}{M!N!} U(L; M, N) \delta_{L,M+N}$$
$$= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \frac{\lambda^{M+N}}{M!N!} [U(L+1; M+1, N) + U(L+1; M, N+1)] \delta_{L,M+N},$$

applied in reverse order to the definition equation (2.12) leads us to the relation

$$T_{ex} = \frac{d}{d\lambda} U_{ex} + 2c U_{ex}. \tag{6.8}$$

It is thus clear that only U_{ex} needs further detailed calculation. In doing so, after straightforward calculation the following intermediate result for U_{ex} is found:

$$\begin{aligned} U_{ex} = {} & 4e^{-2\lambda c} - 6\lambda e^{-2\lambda c} - e^{2\lambda(1/2-c)} \left[(5+\partial^2) + \frac{\lambda}{2}(1-\partial^2)(5-\partial^2) \right. \\ & \left. + \frac{\lambda^2}{4}(1-\partial^2)^2 \right] \partial\alpha(z)|_{z=0} - e^{2\lambda(1/2-c)} \left[(5+\partial^2) + \frac{\lambda}{2}(1+\partial)(1-\partial^2)(5-\partial^2) \right. \\ & \left. + \frac{\lambda^2}{4}(1+\partial)^2(1-\partial^2)^2 \right] \partial\alpha(z)|_{z=\lambda} + e^{2\lambda(1/4-c)} \left[-4(5+\partial^2) \right. \\ & \left. - \lambda\partial(1-\partial^2)(5-\partial^2) + \frac{\lambda^2}{4}(2-\partial^2)(1-\partial^2)^2 \right] \partial\alpha(z)|_{z=\lambda/2}, \end{aligned} \tag{6.9}$$

with

$$\lambda = \kappa^2 \Delta(x).$$

Making use of (4.10) it is possible to express the right-hand side of (6.9) in terms of the modified Struve functions of zeroth and first order to obtain the final result:

$$\begin{aligned} U_{ex} = {} & 4e^{-2\lambda c} - 6\lambda e^{-2\lambda c} - e^{2\lambda(1/2-c)}[-4 - 6\lambda + \tfrac{3}{2}\lambda^2 - \tfrac{3}{2}\pi\lambda(\lambda + \tfrac{1}{2})\mathbf{L}_0(\lambda) + \tfrac{1}{2}\pi\lambda^2 \mathbf{L}_1(\lambda)] \\ & + e^{2\lambda(1/4-c)}[8 + \tfrac{1}{2}\lambda^2 + \tfrac{3}{2}\pi\lambda \mathbf{L}_0(\tfrac{1}{2}\lambda) + \tfrac{1}{4}\pi\lambda^2 \mathbf{L}_1(\tfrac{1}{2}\lambda)]. \end{aligned} \tag{6.10}$$

Furthermore we also deduce

$$b_{ex}=\left(\frac{2}{9}\frac{d}{d\lambda}-\frac{1-4c}{9}\right)a(\lambda),$$

$$c_{ex}=\left(\frac{1}{18}\frac{d}{d\lambda}-\frac{5-2c}{18}\right)a(\lambda),$$

with

$$a(\lambda)=(2-3\lambda+\tfrac{1}{2}\lambda^2)e^{-2\lambda c}+e^{2\lambda(1/2-c)}[2+3\lambda-\lambda^2+\tfrac{3}{2}\pi\lambda(\lambda+\tfrac{1}{2})\mathbf{L}_0(\lambda)-\tfrac{1}{2}\pi\lambda^2\mathbf{L}_1(\lambda)]. \tag{6.11}$$

It is now only a matter of direct substitution of the expressions (6.10) and (6.11) in the right-hand sides of:

$$\begin{aligned}
18A_{ex}&=\left(\frac{13-12c}{8}-\frac{3}{4}\frac{d}{d\lambda}\right)U_{ex}+\frac{7}{4}b_{ex}+2c_{ex}\\
18B_{ex}&=\left(\frac{-3+4c}{8}+\frac{1}{4}\frac{d}{d\lambda}\right)U_{ex}-\frac{5}{4}b_{ex}-4c_{ex}\\
18C_{ex}&=\left(\frac{-3+12c}{8}+\frac{3}{4}\frac{d}{d\lambda}\right)U_{ex}-\frac{9}{4}b_{ex}\\
18D_{ex}&=\left(\frac{-3+12c}{8}+\frac{3}{4}\frac{d}{d\lambda}\right)U_{ex}-\frac{9}{4}b_{ex}\\
18E_{ex}&=\left(\frac{1-4c}{8}-\frac{1}{4}\frac{d}{d\lambda}\right)U_{ex}+\frac{9}{4}b_{ex}
\end{aligned} \tag{6.12}$$

which are a consequence of (2.13), to obtain the superpropagator in closed form. Note that the parameter c must then be expressed in terms of the gauge parameter c'. The superpropagator $\mathcal{T}$ is an entire function of the free propagator $\Delta(x)$ and it is readily seen from the above results that its asymptotic behaviour as $\Delta\to\infty$ is

$$\mathcal{T}\sim\left(\sum_{15p}\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}\eta_{\alpha_5\alpha_6}\right)(\kappa^2\Delta(x))^{3/2}\,e^{2\kappa^2(1-c)\Delta(x)}, \tag{6.13}$$

where the summation is taken over the 15 permutations of the 6 indices specified by the general form (2.2).

All the calculations in the last four sections could be repeated for the higher-order multi-matrix superpropagators mentioned at the end of § 2, although to carry out these computations would become an enormous task. Nevertheless it is possible to predict the leading behaviour of these superpropagators as $\Delta\to\infty$. Indeed, for an arbitrary superpropagator

$$\mathcal{S}=\langle(\exp\kappa\phi(x))_{\alpha\beta},(\exp\kappa\phi(0))_{\alpha_1\beta_1}\ldots(\exp\kappa\phi(0))_{\alpha_k\beta_k}\rangle, \tag{6.14}$$

there will appear in the result a term

$$b_{ex}=\sum_{L=0}^{\infty}\frac{(k\lambda)^L}{L!}b_L$$

from which it follows that

$$\mathscr{S} \sim (\kappa^2 \Delta(x))^{3/2} \, e^{k\kappa^2(1-c)\Delta(x)}, \tag{6.15}$$

as $\Delta \to \infty$.

7. Conclusions

It has been demonstrated that the Ashmore calculus for the evaluation of the superpropagator (1.1) which arises in quantum gravity, can be used to calculate in closed form a whole class of multi-matrix superpropagators. In order to be able to apply the calculus, it is essential to find for each superpropagator an integral representation the integrand of which only contains expressions that are invariant with respect to a similarity transformation. Therefore the method does not directly help to obtain a finite expression for the three-point function

$$\langle |-g(x)|^{\omega} g_{\alpha\beta}(x), |-g(y)|^{\omega} g_{\gamma\delta}(y), |-g(z)|^{\omega} g_{\mu\nu}(z) \rangle, \tag{7.1}$$

and the realistic superpropagator

$$\langle |-g(x)|^{\omega} g_{\alpha\beta}(x) |-g(x)|^{\omega} g_{\gamma\delta}(x), |-g(0)|^{\omega} g_{\mu\nu}(0) |-g(0)|^{\omega} g_{\sigma\tau}(0) \rangle \tag{7.2}$$

arising in electrodynamics where

$$L \sim -\tfrac{1}{4} F_{\mu\nu} F_{\kappa\lambda} g^{\mu\kappa}(x) g^{\nu\lambda}(x).$$

More powerful techniques must be developed to handle these propagators. Nevertheless the detailed calculation of the particular superpropagator (1.9) leads to an important simplification in (7.2). It was indicated at the end of § 2 that the superpropagator (7.2) involves 17 coefficients. This number can be reduced drastically with the help of the result for (1.9), as may be seen from the fact that contractions of the form

$$\sum_{\beta=\gamma} \langle e_{\alpha\beta}^{\kappa\phi(x)} \, e_{\gamma\delta}^{\kappa'\phi(x)}, e_{\mu\nu}^{\kappa''\phi(0)} \, e_{\sigma\tau}^{\kappa'''\phi(0)} \rangle$$

$$\sum_{\gamma=\sigma} \sum_{\beta=\nu} \langle e_{\alpha\beta}^{\kappa\phi(x)} \, e_{\gamma\delta}^{\kappa'\phi(x)}, e_{\mu\nu}^{\kappa''\phi(0)} \, e_{\sigma\tau}^{\kappa'''\phi(0)} \rangle$$

are related directly to (1.9). We are currently investigating the problem of how to extend the formalism to include the important case (7.2). It is also found that the superpropagator (1.9) behaves asymptotically well in the sense given to it by Salam (1974), to say that the gauge parameter c' can always be chosen such that no ambiguities arise as $\Delta(x)$ tends to infinity. From expression (6.15) it is seen that a suitable choice for c' (or c) makes all propagators of the form (6.14) behave well as $\Delta \to \infty$. It is also a striking result that the fractional power of Δ in (6.15) remains unchanged for all values of k. The simple form of expression (6.15) suggests that a method might exist to obtain the asymptotic behaviour of the superpropagators avoiding their calculation in closed form.

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